One-year premium risk and emergence patterns of the ultimate loss based on conditional distributions

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Introduction

Let $(X_1, X_2, \ldots, X_n)$ denote the cumulative payments associated with a given accident year, where X_i denotes the claims paid up to the i-th development year. The random variables $(X_1, \ldots, X_n)$ are dependent.

At each point of time $k = 1, ..., n-1$, the insurer predicts the value of the aggregate claims by calculating $\mathbb{E}[X_n | X_1, ..., X_k]$. The expected value

$$BE_k = \mathbb{E}[X_n | X_1, ..., X_k]$$

is called the best estimate of the ultimate loss.

Introduction

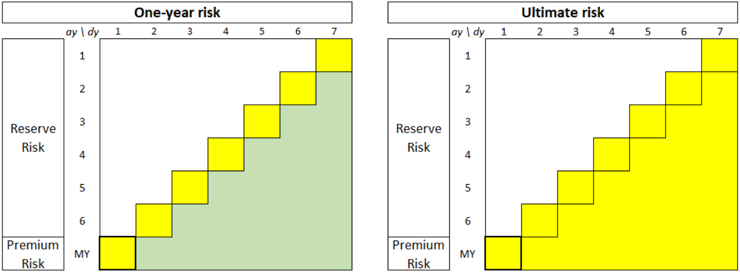
Insurance companies are exposed to premium and reserve risks:

- Premium risk - related to the losses resulting from the premiums that are to be earned in the following year.
- Reserve risk - related to the adequacy of the current volumes of the claims reserves.

We also differ between the notion of ultimate and one-year risk. For premium risk we understand them as the risk that the premiums earned in a given year are not sufficient to cover:

- For ultimate risk the losses paid in an infinite time horizon (the so-called ultimate loss) - described by X_n.
- For one-year risk the losses paid in the first year and the reserve set at the end of the first year - described by BE_1.

Introduction



One-year and ultimate risk perspective for premium and reserve risk based on triangle.

White cells represent historical payments. Yellow cells represent future realized payments. Green cells represent the expected value of future payments conditional on realized payments.

Introduction

For classic reserve risk models, where we know the distribution of X_1 and $X_{i+1}|X_i$, it is clear how to perform a forward simulation of $(X_1, \ldots, X_n)$ and we know the relation between X_n and X_1.

Our first goal and the new problem which we study in this paper is how to model the one-year premium risk and the ultimate premium risk (BE_1, X_n) by generating them in a backward simulation starting with the ultimate loss X_n for the new accident year.

Our second goal is to investigate properties of the one-year risk vs. the ultimate risk, where the risk is measured by Value-at-Risk, in various claims development models.

Introduction

In reserve risk models, so called one-year claims development results *CDRs* are investigated. We want to use a counterpart for the one-year premium risk, which is the technical result for the new accident year defined as the difference between premiums and claims. Since the premiums include an expected profit margin, we replace them with $\mathbb{E}[X_n]$ in our definition:

- $X_n - \mathbb{E}[X_n]$ as the modelled variable for ultimate risk.
- $BE_1 - \mathbb{E}[BE_1]$ as the modelled variable for the one-year risk.

We measure the risk using Value-at-Risk at a given confidence level γ.

Introduction

Motivation:

1. One-year risk needs to be investigated by the companies for Solvency II risk capital (Solvency capital requirement). Many companies have already created models for simulating their ultimate losses and we can modify them into one-year models by means of backward simulation.
2. From business point of view, the unconditional distribution of X_n is well-understood by decision makers, is used in all planning reports, is the basis of pricing, long-term risk analysis and allows for plausibility checks of the results.

The Solvency capital requirement shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5% over a one-year period.

Introduction

We will model the relations between one-year and ultimate premium risk by finding the true emergence pattern, which is defined as the conditional distribution of $BE_1|X_n$.

From the conditional distribution of $BE_1|X_n$, we next derive the unconditional distribution of BE_1 used for quantifying the *true* (unconditional) one-year premium risk. This will allow us to study the true relation between the one-year and ultimate premium risk in models with various distributions of the ultimate loss and various claims development processes.

Emergence pattern

We follow the approach of [England(2012)] and [Bird, Cairns(2011)], who introduce the concept of an emergence pattern of the ultimate loss. They postulate the following relation of BE_1 and X_n by using a simple linear function:

$$BE_1^{ep} = \alpha X_n + (1 - \alpha)\mathbb{E}[X_n],$$

where α is called an emergence factor and $\alpha \in (0, 1)$.

Emergence pattern

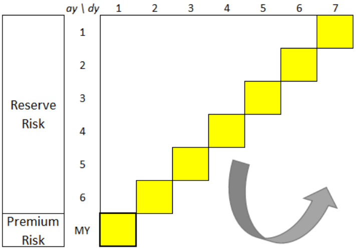
To calculate the α parameter we follow the idea of [England(2012)] and [Bird, Cairns(2011)]

$$\alpha = \frac{SD[BE_1]}{SD[X_n]} = \frac{SD[BE_1 - \mathbb{E}[BE_1]]}{SD[X_n - \mathbb{E}[X_n]]}.$$

The estimation of standard deviation of BE_1 and X_n is done in two steps:

1. We estimate the distribution of development factors $(X_{i+1}|X_i)_{i=1}^{n-1}$ or $(X_{i+1} - X_i)_{i=1}^{n-1}$ from the historical losses in a run-off triangle in a claims reserving model.
2. We estimate the unconditional distribution of X_1 using e.g. an additive model, which is usually related to the planned volume of exposure, coming from financial plans.

Emergence pattern



One-year simulation plus estimation of future payments.

Additional assumption - we do not consider estimation error and we assume that all parameters of the claims development process are given - as a result one-year premium risk can be investigated independently of one-year reserve risk.

Key goal - analysis of probabilistic properties implied by various processes.

Emergence pattern

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mathbb{E}[X_n],$$

Theorem

We have the following properties of the emergence pattern:

1. $\mathbb{E}[BE_1^{ep}] = \mathbb{E}[X_n]$ *and* $Var[BE_1^{ep}] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma\left[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]\right] = \alpha VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right] < VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]$,
3. *If* X_n *has a light-tailed (subexponential with all moments finite) distribution, then* BE_1^{ep} *has a light-tailed (subexponential with all moments finite) distribution,*
4. *If* X_n *has a heavy-tailed distribution with tail index* θ, *then* BE_1^{ep} *has a heavy-tailed distribution with tail index* θ, *and we have the limit*

$$\lim_{\gamma\to 1} \frac{VaR_\gamma\left[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]\right]}{VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]} = \alpha.$$

Emergence pattern

Theorem

1. *The one-year risk is lower than the ultimate risk at all confidence levels,*
2. *The one-year risk decreases linearly in α when the emergence factor α decreases at all confidence levels,*
3. *The distributions of the one-year risk and the ultimate risk have the same tail behavior.*

Disadvantages of the emergence pattern approach:

1. The emergence pattern is true only if BE_1 is perfectly linearly correlated with X_n, i.e. if $\rho(BE_1, X_n) = 1$.
2. The conditional distribution of $BE_1^{ep} | X_n = x$ is degenerate.
3. The true relation between $VaR_\gamma[BE_1]$ and $VaR_\gamma[X_n]$ varies across the models and may not be linear in α. Additionally, it may not be equal to the relation of the $SD[BE_1]$ and $SD[X_n]$.
4. BE_1^{ep} is derived from X_n by a location and scale transformation.

Incremental Loss Ratio Gaussian model

Firstly, we consider Incremental Loss Ratio model with Gaussian incremental losses.

$$X_j = \sum_{i=1}^{j} \epsilon_i, \qquad \text{where:} \quad \epsilon_i \sim N(Em_i, E\sigma_i^2) \quad \text{for } i \in \{1, ..., n\},$$
$$\text{and:} \quad \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1, ..., n\}.$$

E denotes the exposure in the accident year under consideration, and ϵ_i represents the incremental loss in development year i. We will denote

$$m = \sum_{i=1}^{n} m_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model the best estimate of the ultimate loss after the first year is calculated by

$$BE_1 = \mathbb{E}[X_n | X_1] = X_1 + E(m - m_1).$$

Incremental Loss Ratio Gaussian model

Proposition

Let us consider the Incremental Loss Ratio Gaussian model of claims development. We have the following loss distributions:

$$
\begin{aligned}
X_n &\sim N(Em, E\sigma^2), \\
BE_1 &\sim N(Em, E\sigma_1^2), \\
X_1 | X_n = x &\sim N\left(\frac{\sigma_1^2}{\sigma^2}\Big(x - E(m - m_1)\Big) + \frac{\sigma^2 - \sigma_1^2}{\sigma^2} Em_1;\ E\frac{\sigma_1^2(\sigma^2 - \sigma_1^2)}{\sigma^2} \right), \\
BE_1 | X_n = x &\sim N\left(\frac{\sigma_1^2}{\sigma^2} x + \frac{\sigma^2 - \sigma_1^2}{\sigma^2} Em; E\frac{\sigma_1^2(\sigma^2 - \sigma_1^2)}{\sigma^2} \right).
\end{aligned}
$$

Incremental Loss Ratio Gaussian model

We are able to improve the emergence pattern formula so that it yields the correct conditional distribution of $BE_1|X_n$ and unconditional distribution of BE_1 in the reserve risk model and does not depend explicitly on the distributions of X_1 and $X_{i+1}|X_i$ - it depends only on the distribution of X_n and the emergence factor α.

Theorem

Let us set

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \sigma^2_{X_n} = Var[X_n], \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We consider the ILR gaussian model with $X_n \sim N(\mu_{X_n}, \sigma^2_{X_n})$. We have the following distributions of the best estimate of the ultimate loss:

$$\begin{aligned} BE_1|X_n = x &\sim N\Big(\alpha^2 x + (1-\alpha^2)\mu_{X_n}; \alpha^2(1-\alpha^2)\sigma^2_{X_n}\Big), \\ BE_1 &\sim N\Big(\mu_{X_n}, \alpha^2\sigma^2_{X_n}\Big). \end{aligned}$$

Incremental Loss Ratio Gaussian model

Theorem

Let us consider the ILR Gaussian model.

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] = \alpha VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right] < VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]$, *where*

$$VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] = \alpha \Phi^{-1}(\gamma) SD[X_n].$$

Following the emergence pattern formula we have

$$BE_1^{ep} = \alpha X_n + (1-\alpha)\mu_{X_n} \quad \text{and} \quad BE_1^{ep} \sim N\left(\mu_{X_n}, \alpha^2 \sigma^2_{X_n}\right).$$

For this specific model the emergence pattern formula yields the proper unconditional distribution of the one-year risk.

Multiplicative lognormal model

Secondly, we consider a multiplicative loss model where the development factors are modelled with lognormal distributions. We deal with the cumulative payments determined by the reserve risk model:

$$\begin{aligned} X_j &= X_{j-1} \cdot \epsilon_j, && \text{where: } \epsilon_i \sim LogN(m_i, \sigma_i^2) \quad \text{for } i \in \{1, ..., n\}, \\ X_1 &= \epsilon_1 && \text{and: } \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1, ..., n\}. \end{aligned}$$

We will denote

$$m = \sum_{i=1}^{n} m_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

In this reserve risk model the best estimate of the ultimate loss after the first year is calculated as

$$BE_1 = \mathbb{E}[X_n | X_1] = X_1 e^{m - m_1 + \frac{1}{2}(\sigma^2 - \sigma_1^2)}.$$

Multiplicative lognormal model

Theorem

For $X_n \sim LogN$ with the expected value μ_{X_n} and variance $\psi^2_{X_n}\mu^2_{X_n}$

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \psi_{X_n} = \frac{SD[X_n]}{\mathbb{E}[X_n]}, \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We receive the distributions of the best estimate of the ultimate loss:

$$BE_1|X_n = x \sim LogN\left(\tilde{\alpha}^2 \log(x) + \left(1-\tilde{\alpha}^2\right)\left(\tilde{m} + \frac{\tilde{\sigma}^2}{2}\right); \tilde{\alpha}^2(1-\tilde{\alpha}^2)\tilde{\sigma}^2\right),$$

$$BE_1 \sim LogN\left(\tilde{m} + (1-\tilde{\alpha}^2)\frac{\tilde{\sigma}^2}{2}; \tilde{\alpha}^2\tilde{\sigma}^2\right),$$

where the parameters are

$$\tilde{m} = log(\mu_{X_n}) - \frac{1}{2}\log\left(1+\psi^2_{X_n}\right), \quad \tilde{\sigma}^2 = \log\left(1+\psi^2_{X_n}\right),$$

$$\tilde{\alpha}^2 = \frac{\log\left(1+\alpha^2\psi^2_{X_n}\right)}{\log\left(1+\psi^2_{X_n}\right)}.$$

Multiplicative lognormal model

Theorem

Let us consider the multiplicative lognormal model.

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] < VaR_\gamma[X_n - \mathbb{E}[X_n]]$ *for* $\gamma > \gamma^*$,
   $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] > VaR_\gamma[X_n - \mathbb{E}[X_n]]$ *for* $\gamma < \gamma^*$, *where*

$$\begin{aligned} VaR_\gamma[BE_1] &= \mu_{X_n}\left(1+\alpha^2\psi_{X_n}^2\right)^{-1/2} e^{\sqrt{\log\left(1+\alpha^2\psi_{X_n}^2\right)}\Phi^{-1}(\gamma)}, \\ \gamma^* &= \Phi\left(\frac{1}{2}\sqrt{\log(1+\alpha^2\psi_{X_n}^2)} + \frac{1}{2}\sqrt{\log(1+\psi_{X_n}^2)}\right), \end{aligned}$$

3. *We have the limit*

$$\lim_{\gamma\to 1} \frac{VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right]}{VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]} = 0.$$

Multiplicative lognormal model

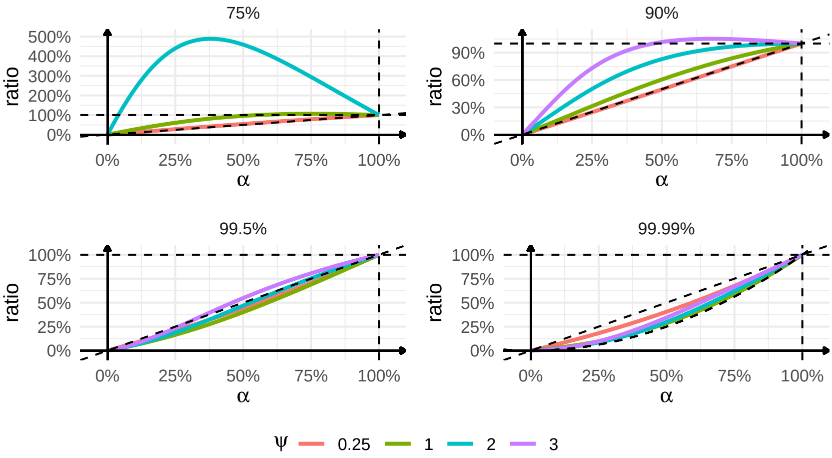
Following the emergence pattern formula we have

$$\begin{aligned} BE_1^{ep} &\sim \alpha \cdot LogN(\tilde{m}, \tilde{\sigma}^2) + (1-\alpha)\mu_{X_n} \\ BE_1 &\sim LogN\left(\tilde{m} + (1-\tilde{\alpha}^2)\frac{\tilde{\sigma}^2}{2}; \tilde{\alpha}^2\tilde{\sigma}^2\right) \end{aligned}$$

For this model the emergence pattern formula has a shifted lognormal distribution and does not yield the proper unconditional distribution of the one-year risk.

The relation between $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right]$, $VaR_\gamma\left[BE_1^{ep} - \mathbb{E}[BE_1^{ep}]\right]$ and $VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]$ for multiplicative lognormal model is analyzed in more detail in [Szatkowski(2020)].

Multiplicative lognormal model



The ratios $\frac{VaR_\gamma[BE_{\mathbf{1}} - \mathbb{E}[BE_{\mathbf{1}}]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the Hertig's Lognormal model.

Over-Dispersed Poisson model

We finally consider Incremental Loss Ratio model with Over-Dispersed Poisson incremental losses. We investigate the cumulative payments determined by the reserve risk model:

$$X_j = \sum_{i=1}^{j} \epsilon_i, \qquad \text{where:} \quad \epsilon_i \sim ODP\big(\mu\omega_i, \psi\big) \quad \text{for } i \in \{1, ..., n\},$$

$$\text{and:} \quad \epsilon_i \perp\!\!\!\perp \epsilon_j \quad \text{for } i \neq j \in \{1, ..., n\}.$$

We assume that $\mu > 0, \omega_i > 0, \psi > 0$. The assumption that $\epsilon \sim ODP\big(\mu, \psi\big)$ means that $\epsilon/\psi \sim Poiss\big(\mu/\psi\big)$. We assume

$$1 = \sum_{i=1}^{n} \omega_i.$$

In this reserve risk model the best estimate of the ultimate loss after the first year is calculated by

$$BE_1 = \mathbb{E}[X_n | X_1] = X_1 + \mu(1 - \omega_1).$$

Over-Dispersed Poisson model

Theorem

Let us set

$$\mu_{X_n} = \mathbb{E}[X_n], \quad \psi_{X_n} = \frac{Var[X_n]}{\mathbb{E}[X_n]}, \quad \alpha = \frac{SD[BE_1]}{SD[X_n]}.$$

We consider the Over-Dispersed Poisson model. We have the following loss distributions:

$$X_n \sim \psi_{X_n} \cdot Poiss\left(\mu_{X_n}/\psi_{X_n}\right), \tag{0.1}$$

$$BE_1 | X_n = x \sim \psi_{X_n} \cdot Bin\left(\frac{x}{\psi_{X_n}}; \alpha^2\right) + (1-\alpha^2)\mu_{X_n}, \tag{0.2}$$

$$BE_1 \sim \psi_{X_n} \cdot Poiss\left(\alpha^2 \mu_{X_n}/\psi_{X_n}\right) + (1-\alpha^2)\mu_{X_n}, \tag{0.3}$$

with the parameter α which satisfies $\alpha \in (0,1)$.

Over-Dispersed Poisson model

Theorem

Let us consider the Over-Dispersed Poisson model.

1. $\mathbb{E}[BE_1] = \mathbb{E}[X_n]$ *and* $Var[BE_1] = \alpha^2 Var[X_n] < Var[X_n]$,
2. $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] < VaR_\gamma[X_n - \mathbb{E}[X_n]]$ *for* $\gamma > \gamma^*$ *where* $\gamma^* < 1$ *is the last point where the distribution functions of* BE_1 *and* X_n *intersect, or* $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right] < VaR_\gamma[X_n - \mathbb{E}[X_n]]$ *for all* γ *if the distribution functions of* BE_1 *and* X_n *don't intersect. The relation between* $VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right]$ *and* $VaR_\gamma[X_n - \mathbb{E}[X_n]]$ *can change for* $\gamma < \gamma^*$ *since the distribution functions of* BE_1 *and* X_n *can intersect more than once (if they intersect),*
3. *We have the limit*

$$\lim_{\gamma \to 1} \frac{VaR_\gamma\left[BE_1 - \mathbb{E}[BE_1]\right]}{VaR_\gamma\left[X_n - \mathbb{E}[X_n]\right]} = 1.$$

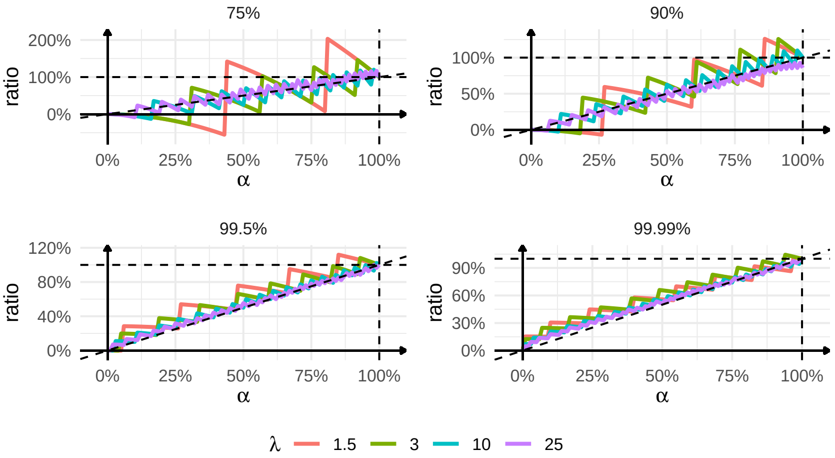
Over-Dispersed Poisson model

Following the emergence pattern formula we have

$$
\begin{aligned}
BE_1^{ep} &\sim \alpha\psi_{X_n} \cdot Poiss\left(\mu_{X_n}/\psi_{X_n}\right) + (1-\alpha)\mu_{X_n}, \\
BE_1 &\sim \psi_{X_n} \cdot Poiss\left(\alpha^2\mu_{X_n}/\psi_{X_n}\right) + (1-\alpha^2)\mu_{X_n}.
\end{aligned}
$$

For this model the emergence pattern formula has a shifted Poisson distribution with different parameters and does not yield the proper unconditional distribution of the one-year risk.

Over-Dispersed Poisson model



The ratios $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the Poisson model.

Emergence pattern - arbitrary X_n

Our next step is to modify the reparametrized approach, so that we may use an arbitrary distribution of the ultimate loss. What we suggest is to keep the conditional distribution of $BE_1|X_n$ and use any unconditional distribution of the ultimate loss X_n.

Firstly, we have a flexible and interpretable probabilistic model, where we can switch from the ultimate risk to the one-year risk and which can be used in Solvency II premium risk modelling.

Secondly, we can investigate properties of the one-year risk vs. the ultimate risk in various claims development models, beyond the models we know from the claims reserving literature.

Emergence pattern - arbitrary X_n

For the ILR Gaussian model we have the following representation:

$$BE_1 = \alpha^2 X_n + (1-\alpha^2)\mu_{X_n} + \sqrt{\alpha^2(1-\alpha^2)}\sigma_{X_n}\xi,$$

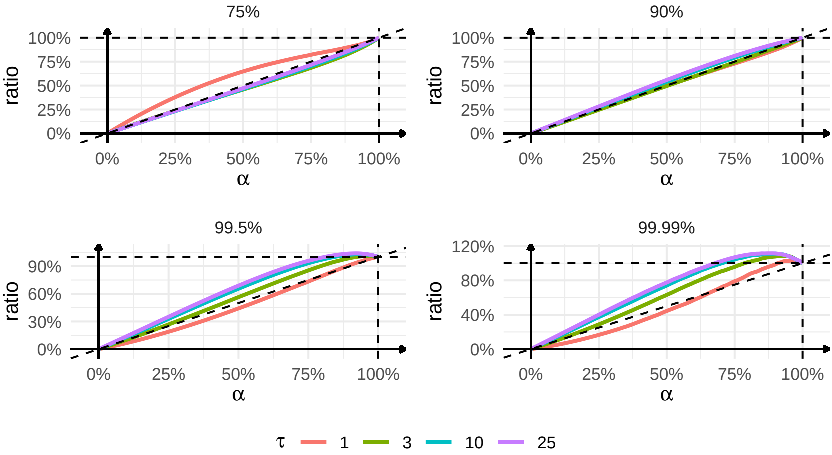
where $\xi \sim N(0,1), X_n \sim N(\mu_{X_n}, \sigma^2_{X_n})$, and ξ is independent of X_n.
It can be seen as an extension of the classical emergence pattern formula, where we simply add a Gaussian noise in order to have a non-degenerate distribution of $BE_1|X_n = x$.

For the multiplicative LN model we have the following representation:

$$BE_1 = (X_n)^{\tilde{\alpha}^2} e^{(1-\tilde{\alpha}^2)\left(\tilde{m}+\frac{\tilde{\sigma}^2}{2}\right)} e^{\sqrt{\tilde{\alpha}^2(1-\tilde{\alpha}^2)}\tilde{\sigma}\xi},$$

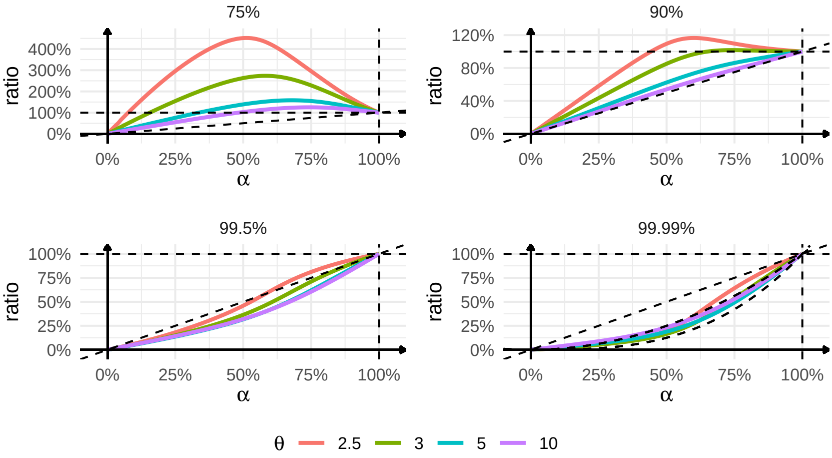
where $\xi \sim N(0,1), X_n \sim LogN(m, \sigma^2)$, and ξ is independent of X_n.
It can be seen as an extension of the classical emergence pattern formula, where we allocate the ultimate loss X_n to BE_1 with a random scaling factor in order to have a non-degenerate distribution of $BE_1|X_n = x$.

Multiplicative lognormal model - arbitrary X_n



The ratios $\frac{VaR_\gamma[BE_\mathbf{1} - \mathbb{E}[BE_\mathbf{1}]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the model where $X_n \sim Weibull(\tau)$ and $BE_1|X_n$ comes from the Hertig's Lognormal model.

Multiplicative lognormal model - arbitrary X_n



The ratios $\frac{VaR_\gamma[BE_1 - \mathbb{E}[BE_1]]}{VaR_\gamma[X_n - \mathbb{E}[X_n]]}$ in the model where $X_n \sim Pareto(\theta)$ and $BE_1|X_n$ comes from the Hertig's Lognormal model.

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Dziękuję bardzo za uwagę.

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